

Q.1 Define order of an element of a group with an example.

Soln: Order of an element of a group:

Let  $G$  be a group under multiplication. Let  $e$  be the identity element in  $G$ . Let us suppose  $a$  is any element of  $G$  then the least positive integer  $n$ , if exist, such that  $a^n = e$  is said to be order of an element  $a \in G$ , and can be written as  $O(a) = n$ .

In case, such a positive integer  $n$  does not exist, we say that the element  $a$  is of infinite or zero order.

For example: In the multiplicative group  $Q_0$  of non-zero rational numbers  $1, -1 \in Q_0$  such that

$O(1) = 1$  and  $O(-1) = 2$ . The order of any other element of this group is infinite.

Q.2 Define integral power of an element.

Soln: Integral Power of An Element:

Let  $G$  be a group with respect to multiplication. If  $a \in G$ , then  $a^a$  is denoted by  $a^2$ ,  $a^{aa}$  is denoted by  $a^3$  and so on. We have  $a^{aa} \dots$   $n$  times  $= a^n$ ,  $n \in \mathbb{Z}^+$ .

But closure property  $a^2, a^3, \dots, a^n \in G$ .

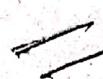
Also, if  $e$  is the identity element in  $G$ , we define  $a^0 = e$ .

If  $n$  is a positive integer, we define

$$\bar{a}^n = (a^n)^{-1} \in G \quad [ \because a^n = a \cdot a \dots n \text{ times } \in G ]$$

$$\text{Further, } (a^n)^{-1} = (a \cdot a \dots n \text{ times})^{-1} = \bar{a} \cdot \bar{a} \dots n \text{ times} \\ = (\bar{a})^n$$

$$\text{Thus, } \bar{a}^n = (a^n)^{-1} = (\bar{a})^n.$$



Q. 3. Consider the multiplicative group

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$$G = \{1, -1, i, -i\}$$

of cube roots of unity. Find the order of each element of  $G$ .

Soln: Since, 1 is the identity element, therefore

$$o(1) = 1$$

$$\text{Also, } (-1)^2 = 1 \Rightarrow o(-1) = 2$$

$$(i)^4 = 1 \Rightarrow o(i) = 4$$

$$(-i)^4 = 1 \Rightarrow o(-i) = 4$$

Q. 4. The order of every element of a finite group is finite.

Proof: Let  $G$  be a finite multiplicative group and  $a \in G$ .

Let us consider all positive integral powers of  $a$ . i.e.

$$a, a^2, a^3, \dots, a^k, \dots, a^\infty \dots$$

By closure property, these all are elements of  $G$ .  
Since,  $G$  is finite, therefore, all the integral power of  $a$  can not be distinct elements of  $G$ .

Let us suppose that  $a^k = a^l$ , where  $k > l$  — (1)

$$\text{Then, } a^k = a^l \Rightarrow a^k \cdot a^{-l} = a^l \cdot a^{-l}$$

$$\Rightarrow a^{k-l} = a^{l-k}$$

$$\Rightarrow a^{k-l} = a^0 = e$$

$$\Rightarrow a^m = e, \text{ where } m = k - l > 0$$

Thus, there exist a positive integer  $m$  such that  $a^m = e$ . Now since every set of positive integers has a least member it follows that the set of all positive integers  $m$  such that  $a^m = e$  has a least member say  $n$ .

Thus,  $o(a) = n$ , which is finite.

Hence, the order of every element of the finite group  $G$  is finite.

Proved.