B.Sc-part-I(H), paper-I Abstract Algebra Group. 17-05-2020

Q.1 Define order of an element of a Group with an example.

Soln: Order of an element of a group:-

Let G be a group under multiplication. Let e be the identity element in G. Let us suppose a is any element of G then the least positive integer n, if exist, such that $a^n = e$ is said to be order of an element $a \in G$, and can be written as

$$o(a) = n.$$

In case, such a positive integer n does not exist, we say that the element a is of infinite or zero order.

For example. In the multiplicative group Q_0 of non-zero rational number $1, -1 \in Q_0$ such that

$o(1) = 1$ and $o(-1) = 2$. The order of any other element of this group is infinite.

Q.2. Define integral power of an element.

Soln. Integral Power OF An Element.

Let G be a group with respect to multiplication. If $a \in G$, then aa is denoted by a^2, aaa is denoted by a^3 and so on. We have

$aaa \ldots n$ times $= a^n$, $n \in Z^+$.

$aaa \ldots \ldots a^n \in G$.

But closure property $a^2, a^3, \ldots \ldots a^n \in G$. We ~~define~~

Also, if e is the identity element in G, we define $a^0 = e$.

If n is a positive integer, we define

$a^{-n} = (a^n)^{-1} \in G$ [$\because a^n = aaa \ldots n$ times $\in G$]

Further, $(a^n)^{-1} = (aa \ldots n \text{ times})^{-1} = a^{-1} a^{-1} \ldots n$ times $= (a^{-1})^n$

Thus, $a^{-n} = (a^n)^{-1} = (a^{-1})^n$.

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Q.3. Consider the multiplicative group

$G = \{1, -1, i, -i\}$

of cube roots of unity. Find the order of each element of G.

Soln: Since, 1 is the identity element, therefore

$o(1) = 1$

Also, $(-1)^2 = 1 \Rightarrow o(-1) = 2$

$(i)^4 = 1 \Rightarrow o(i) = 4$

$(-i)^4 = 1 \Rightarrow o(-i) = 4$

Q.4 The order of every element of a finite group is finite.

Proof: Let G be a finite multiplicative group and $a \in G$.

Let us consider all positive integral powers of a. i.e.

$a, a^2, a^3, \ldots a^s, \ldots a^r \ldots\ldots$

By closure property, these all are element of G.

Since, G is finite, therefore, all the integral power of a can not be distinct element of G

Let us suppose that $a^r = a^s$, where $r > s$ — (1)

Then, $a^r = a^s \Rightarrow a^r a^{-s} = a^s \cdot a^{-s}$

$\Rightarrow a^{r-s} = a^{s-s}$

$\Rightarrow a^{r-s} = a^0 = e$

$\Rightarrow a^m = e$, where $m = r-s > 0$

Thus, there exist a positive integer m such that $a^m = e$. Now since every set of positive integers has a least member it follows that the set of all positive integer m such that $a^m = e$ has a least member say n.

Thus, $o(a) = n$, which is finite.

Hence, the order of every element of the finite group G is finite.

Proved.